TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 264, Number 2, April 1981

THE RADON-NIKODYM PROPERTY IN CONJUGATE BANACH SPACES. II

BY

CHARLES STEGALL

ABSTRACT. In the first part of this article the following result was proved.

THEOREM. *The dual of a Banach space X has the Radon-Nikodym property if and only if for every closed, linear separable subspace Y of X, Y^* is separable. We find other, more detailed descriptions of Banach spaces whose duals have the Radon-Nikodym property.*

0. Introduction. This paper extends and applies the results of **[26]**. We frequently use the results and proofs of **[26]**, particularly the following.

THEOREM 0. *If X is a Banach space, then X^* has the Radon-Nikodym property (RNP) if (and only if) every separable, linear subspace of X has a separable dual.*

We extend this result to obtain, among other things, that X^* has RNP if and only if X is an Asplund space (what Asplund **[1]** called a "Strong differentiability space"). This result was obtained in **[25]** by combining results of **[1]**, **[16]**, **[17]** and **[19]**–**[22]**. The proof given here is more direct. Indeed, we prove here a stronger result from which this result follows. Our notation and terminology are that of **[26]**.

In **[26]** criteria are given for an operator (a continuous linear function) to be a factor of "the Haar operator." The Haar operator is defined to be any operator from l_1 (the space of absolutely summing sequences) into $L_\infty(\Delta, \mu)$ (the L_∞ space of the Haar measure μ on the Cantor space Δ) that carries the basis vectors of l_1 onto the usual (Haar) basis vectors of $C(\Delta)$ (considered a subspace of $L_\infty(\Delta, \mu)$), see **[26]**. We prove here that if T is *not* a factor of the Haar operator, then it factors through an Asplund space. This is done in §2.

Motivated by a definition of Grothendieck **[6]**, in §1 we define a class of sets (the GSP sets) which have very nice permanence properties, and are dual in some sense to RNP sets. Roughly speaking, a bounded subset of a Banach space is a GSP set if X^* has RNP, or, if X^* does not have RNP, one cannot make the construction of **[26]** with the "Haar system" inside S. This is made precise in §§1 and 2. The most important examples of GSP and RNP sets are the weakly compact sets.

In §2, we give, modulo some well-known results about differentiability, a lemma of Phelps and the results of **[26]**, a self-contained proof of a "Geometric Lemma" from which the results in this section follow immediately.

In §3, we state a result, although apparently not too general, which does seem to

Received by the editors October 19, 1977 and, in revised form, April 1, 1980.
1980 *Mathematics Subject Classification.* Primary 46B22; Secondary 46G10.

include the most interesting known cases: e.g. a WCG Banach space is a weak Asplund space, a result proved by Asplund [1] using quite different techniques.

§4 consists of examples and some questions.

We assume, only for simplicity, that our Banach spaces are real Banach spaces.

Most of these results were obtained while the author was a member of the Sonderforschungsbereich 72 der Universität Bonn.

The author would like to thank J. Diestel and R. R. Phelps for reading and commenting on a preliminary version of this paper.

1. RNP and GSP sets and their duality.

DEFINITION 1.1. Let K be a closed, bounded, convex subset of a Banach space X. We say that K has the Radon-Nikodym property if for any finite measure space (Ω, Σ, μ), any $m: \Sigma \to X$ that is μ-continuous, countably additive, of finite variation, with average range $\{\mu(E)^{-1}m(E): E \in \Sigma, \mu(E) > 0\}$ contained in K, is representable by a Bochner integrable function. That is, there exists $f: \Omega \to X$, Borel measurable, essentially separably valued, with $m(E) = \int_E f\, dP$. The function f is called the derivative of m. It is easy to see that $f(\omega) \in K$ a.e.

A Banach space has RNP if every closed, bounded and convex subset has RNP.

The most important examples of RNP sets are the weakly compact sets. This can be proved in several ways; perhaps the easiest is [2].

We state the following theorem, due to a number of authors–see [4] for a discussion of RNP.

THEOREM. *Let K be a closed, bounded, convex subset of a Banach space. Then the following are equivalent.*

(1) *K is an RNP set;*

(2) *every closed (convex) subset of K is dentable;*

(3) *every closed, convex subset of K is the closed, convex hull of its strongly exposed points.*

The most remarkable part of this remarkable theorem is that (2) implies (3), which is due to Bourgain (in [2] it is proved for weakly compact sets–the arguments given there easily generalize). We do not need this theorem so we shall not go into the details here (see [2] and, for a survey, [4]). However, we need to know that an element x^* in X^* exposes a subset K of X if x^* attains its supremum on K, and does so at only one point in K; and $x^* \in X^*$ strongly exposes K if it attains its supremum, say at x_0 in K, and if $x^*(x_n)$ converges to $x^*(x_0)$, x_n in K, then $\|x_n - x_0\|$ converges to zero.

We mention the special case of RNP sets we shall be interested in: where K is an RNP subset of some conjugate Banach space and is weak* closed (hence weak* compact).

If K is a convex, weak* compact and convex subset of X^*, then K is an RNP set if and only if for every Radon measure μ on $\mathfrak{B}$ (the weak* Borel subsets of K), the resolvent mapping m from $\mathfrak{B}$ to X^* given by

$$m(B)(x) = \int_B x^*(x)\, d\mu(x^*)$$

is representable by a Bochner integrable function. This is equivalent to saying that μ is supported a.e. on a countable union of norm compact sets. Details about this can be found in [**6**], [**26**] and [**4**].

We shall repeatedly use the following fact: if $K_i \subseteq X_i$, $i = 1, 2$, then K_1 and K_2 are RNP sets if and only if $K_1 \times K_2$ is an RNP subset of $X_1 \times X_2$. The case we shall use very often is that K is an RNP set if and only if $K \times I$ is an RNP subset of $X \times \mathbf{R}$ where I is any closed, bounded interval. This is a complicated way of stating the usual real-valued version of the Radon-Nikodym theorem.

If $\{x_i: i \in I\}$ is a family of Banach spaces indexed by the set I, then $(\Sigma_I X_i)_p$, $1 \leqslant p < +\infty$, and $(\Sigma_I X_i)_0$ denote the usual l_p and c_0 sums. (See [**26**].)

A Banach space X is always considered as canonically embedded in its bidual X^{**}.

If x is an element of X and $\varepsilon > 0$, we denote by $B_X(x, \varepsilon)$ (resp. $\bar{B}_X(x, \varepsilon)$) the open (resp. closed) ball in X with center x and radius ε.

If K is a compact Hausdorff space and μ a Radon measure on K, then $C(K)$, $L_\infty(K, \mu)$ and $L_1(K, \mu)$ denote the Banach spaces of continuous functions on K and the L_∞ and L_1 spaces, respectively. More generally, (Ω, Σ, μ) will denote a general positive measure space and $L_p(\Omega, \Sigma, \mu)$ the usual spaces.

Definition 1.2. (i) A subset S of $L_\infty(\Omega, \Sigma, \mu)$ is said to be equimeasurable [**6**] if for every $\varepsilon > 0$ there exists a set $B \in \Sigma$, $\mu(B) > \mu(\Omega) - \varepsilon$, and $\{f|_B: f \in S\}$ is a relatively compact subset of $L_\infty(\Omega, \Sigma, \mu)$.

(ii) A bounded subset S of a Banach space X is said to be a GSP set if for any finite measure spaces (Ω, Σ, μ) and any continuous, linear function T: $X \to L_\infty(\Omega, \Sigma, \mu)$, the set $\{Tx: x \in S\}$ is equimeasurable.

Suppose S is a closed, bounded and convex subset of the Banach space X. Then, rephrasing Proposition 9 of [**6**], we have that S is a GSP set if and only if the following holds:

For every Banach space Y, every probability space (Ω, Σ, μ), every T: $Y \to X$ with $T(\bar{B}_Y(0, 1))$ contained in S and every I: $X \to L_\infty(\Omega, \Sigma, \mu)$ then JIR is a nuclear operator where J: $L_\infty(\Omega, \Sigma, \mu) \to L_1(\Omega, \Sigma, \mu)$ is the canonical operator (see [**6**, pp. 78–88 and 141–148]).

This allows a proof of the following. Suppose T: $X \to C(K)$ is an into isomorphism and S is a bounded subset of X. Then, S is a GSP subset of X if and only if $T(S)$ is a GSP subset of $C(K)$. Suppose we have operators T_1: $Z \to X$ with $T(\bar{B}_Z(0, 1)) \subseteq S$ and T_2: $X \to L_\infty(\Omega, \Sigma, \mu)$. Since T is an isomorphism and $L_\infty(\Omega, \Sigma, \mu)$ is an injective space there exists an operator R: $C(K) \to L_\infty(\Omega, \Sigma, \mu)$ such that $RT = T_2$. If J denotes the canonical operator from $L_\infty(\Omega, \Sigma, \mu)$ to $L_1(\Omega, \Sigma, \mu)$ then JR is an integral operator [**6**, loc. cit.] and there exists a positive Radon measure ν on K such that $\int|Rf|\, d\mu \leqslant \int|f|\, d\nu$ for all $f \in C(K)$ (see [**6**, pp. 154–164] and [**11**]). If $T(S)$ is a GSP set then J_1TT_1 is nuclear where T_1: $C(K) \to L_1(K, \nu)$ is canonical. Clearly, JT_ZT_1 is nuclear and we have shown that S is a GSP set.

In the following lemmas S denotes a bounded subset of the Banach space X.

Lemma 1.3. *S is a GSP set if and only if the smallest closed, absolutely convex subset of X containing S is GSP.*

LEMMA 1.4. *The collection of GSP subsets of X forms a hereditary ring and the sum of two GSP sets is a GSP set.*

LEMMA 1.5. *S is GSP if and only if every separable subset of S is GSP.*

LEMMA 1.6. *If $\{S_n\}$, $n = 1, 2, \ldots,$ is a sequence of GSP subsets of X, $\{\lambda_n\}$ is a sequence of positive numbers converging to zero, then $\bigcap_{n=1}^{\infty} S_n + B(0, \lambda_n)$ is GSP.*

LEMMA 1.7. *If T: $X \to Y$ is a bounded linear operator and S is a GSP set, so is $T(S)$.*

LEMMA 1.8. *If T: $X \to Y$ is an into isomorphism, then S is a GSP set if and only if $T(S)$ is a GSP set.*

The proofs of Lemmas 1.3, 1.4, 1.5 and 1.7 are obvious. Lemma 1.6 follows from a simple compactness argument. Lemma 1.8 follows from the remark after Definition 1.2 by considering Y as isometrically embedded into some $C(K)$ space.

LEMMA 1.9. *Let $\{X_n\}$ be a sequence of Banach spaces and S a bounded subset of $X = (\Sigma_{n=1}^{\infty} X_n)_0$. Then S is a GSP set if and only if $P_n(S)$ is a GSP set for every n, where P_n denotes the canonical operator from X to X_n.*

PROOF. One direction is immediate from Lemma 1.7. Assume X_n is isometrically embedded in $C(K_n)$. Let K' be the one point compactification of the disjoint union of the K_n's. Then we may regard $(\Sigma_n X_n)_0$ as a subspace of $C(K')$. Let μ be a Radon measure on K' and $\varepsilon > 0$. Choose m such that $|\mu|(\bigcup_{n>m} K_n) < \varepsilon/2$. By assumption $\{x|_{K_n}: x \in S\}$ is equimeasurable for each n, so is $\{x|_{\bigcup_{n<m} K_n}: x \in S\}$ (the finite sum of such sets). This completes the proof by Lemmas 1.4 and 1.7.

We shall need the following

PROPOSITION 1.10. *If $K \subseteq X^*$, K weak* compact convex, then K is RNP if and only if for any Y separable, T: $Y \to X$, $T^*(K)$ is a norm separable subset of Y.*

PROOF. This is an application of the results of [**26**]. If, for some T and separable Y, $T^*(K)$ is not norm separable, then there exist a subset Δ of $T^*(K)$, Δ homeomorphic to the Cantor set (in the weak* topology), a Haar system $\{h_{n,i}\}$ in Δ and a bounded sequence $\{x_{n,i}\}$ in Y such that

$$\sum_{n>0} \sum_{0<i<2^n} \|Rx_{n,i} - h_{n,i}\| < \varepsilon < 1$$

[**26**, Corollary 1]. Let K_0 be $(T^*)^{-1}(\Delta) \cap K$. Let μ be the measure on Δ such that $\int h_{n,i}\, d\mu = 2^{-n}$ [**26**, Theorem 2]. Regarding μ as a continuous linear function on $C(\Delta)$ and $C(\Delta)$ as a subspace of $C(K_0)$ we may extend μ to a measure ν on K_0 and hence to all of K. By the remarks at the beginning of this section and the results of [**26**] K does not have RNP. Conversely, suppose $T^*(K)$ is separable for all T and all Y separable such that T: $Y \to X$ is an operator. Let K_0 be a norm separable and convex subset of K and Y a separable subspace of X that norms the closed, linear span of K_0. Let T: $Y \to X$ be the containment operator: by hypothesis $T^*(K)$ is separable and T^* is an isometry on K_0. The classical result of Dunford-Pettis-Phillips (see [**26**, p. 213]) gives that $T^*(K)$ is an RNP set. Hence, any measure with averaged range in K_0 must have a derivative.

Let K be a weak* compact, but not necessarily convex, subset of X^*. Let Y be a separable Banach space and T: $Y \to X$. Suppose $T^*(K)$ is norm separable. Then the weak* closed convex hull of $T^*(K)$ in X^* is also norm separable; one need only observe that, in this case, $T^*(K)$ is Polish in both the weak* and norm topologies. Actually, this is a special case of a much more general result due to Haydon **[8]**. By the weak* continuity of T^*, the image of the weak* closed convex hull of K is separable. *The point is that a weak* compact subset K of X* is RNP if and only if the weak* closed, convex hull of K is RNP.* Also, if K_i are weak* compact RNP sets for $1 \leqslant i \leqslant n$, and $a \leqslant t \leqslant b$ then the smallest weak* closed set containing $\bigcup_{1 \leqslant i \leqslant n,\, a \leqslant t \leqslant b} tK_i$ is also a weak* compact RNP set.

COROLLARY 1.11. *If* T: $X \to Y$ *and* K *is a weak* compact RNP subset of* Y^* *then* $T^*(K)$ *is an RNP set.*

We are now prepared to prove the following.

THEOREM 1.12. *Let* T: $X \to Y$ *be an operator. Then* $T(\bar{B}_X(0, 1))$ *is GSP if and only if* $T^* \bar{B}_{Y^*}(0, 1)$ *is RNP.*

PROOF. Let Z be a separable Banach space and R: $Z \to X$ an operator and we may assume $\|R\| = 1$. If $R^* T^*(\bar{B}_{Y^*}(0, 1))$ is not norm separable then we may proceed exactly as in Proposition 1.10 and produce a probability space (Ω, Σ, μ) and an operator U: $Y \to L_\infty(\Omega, \Sigma, \mu)$ such that $JUTR$ is not nuclear and, thus, that $T(\bar{B}_X(0, 1))$ is not GSP (see the remark after Definition 1.2). For the converse, let U: $Y \to L_\infty(\Omega, \Sigma, \mu)$ be an operator. We must show that JUT is nuclear (where J always denotes the canonical operator). Assuming that $\|U\| = 1$; this is equivalent to $(JUT)^*$ being nuclear, which it is, because the vector measure associated with $(JUT)^*$ has average range in $T^*(\bar{B}_{Y^*}(0, 1))$ which is an RNP set (again, see **[6**, pp. 78–88 and 141–148**]**).

THEOREM 1.13. *Every bounded subset of* X *is GSP if and only if* X^* *has RNP.*

Our principal result in this section is the following.

THEOREM 1.14. *An operator* T: $X \to Y$ *transforms bounded sets into GSP sets (equivalently* T^* *transforms bounded sets into RNP sets) if and only if there exists a Banach space* Z *such that* Z^* *has RNP, operators* T_1: $X \to Z$, T_2: $Z \to Y$ *and* $T_2 T_1 = T$.

PROOF. Assume T: $X \to Y$ transforms bounded sets into GSP sets. Let S be the norm closure of $\{Tx: \|x\| \leqslant 1\}$ and $S_n = 2^n S + B(0, 2^{-n})$, $n = 1, 2, \ldots$. We now apply Lemmas 1.6 and 1.8 and the construction of **[3]**.

Let Y_n be the Banach space in which the underlying vector space is Y and the unit ball of Y_n is S_n, i.e. renorm Y with the Minkowski functional of S_n. Obviously, Y_n is isomorphic to Y. Let Y_0 be that subspace of the product $\prod_{n=1}^\infty Y_n$ such that $(y_n) \in Y_0$ if and only if $\lim_n \|y_n\|_n = 0$. That is, Y_0 is the c_0 sum of the Y_n's. Let $Z \subseteq Y_0$, $(y_n) \in Z$ if and only if $y_1 = y_2 = \cdots = y_n$, i.e. (y_n) is a constant sequence. Define T_2: $Z \to Y$ by $T_2((y_i)) = y_1$ and T_1: $X \to Z$ by $T_1(x) = ((Tx))$. If $\|x\| \leqslant 1$, $Tx \in S$, then $2^n Tx \in S_n$ and $\|Tx\|_n \leqslant 2^{-n}$. Thus T_1 is well defined.

(This is exactly the construction of [3].) We need only observe that Z^* has RNP, or equivalently the unit ball of Z is GSP. By Lemma 1.8 we need only know that the unit ball of Z (regarded as a subspace of Y_0) is a GSP subset of Y_0. By Lemma 1.9, we need only observe that the y_n coordinates of Z are GSP. The y_n coordinates of Z are contained in $\bigcap_{n=1}^{\infty} S_n$, which is GSP.

COROLLARY 1.15. *If $S \subseteq X$, then S is GSP if and only if there exists a Banach space Z such that Z^* has RNP and an operator T: $Z \to X$ such that $S \subseteq \{Tz: \|z\| \leqslant 1\}$.*

PROOF. One direction is from Lemma 1.7 and Theorem 1.14. Suppose S is GSP; we may also assume that it is closed, convex and symmetric. Define T_0: $l_1(S) \to X$ by $T_0((\lambda_x)) = \sum_{x \in S} \lambda_x x$. Since T_0 maps bounded sets onto GSP sets, by Theorem 1.14 there exist operators T_1, T_2, and a Banach space Z with Z^* having RNP, T_1: $l_1(S) \to Z$, T_2: $Z \to X$, and $T_0 = T_2 T_1$. Let $T = T_2$.

COROLLARY 1.16. *Let $K \subseteq X^*$ be a weak* compact, convex RNP set. Then there exists a Banach space Z with Z^* having RNP and an operator T: $X \to Z$ with dense range such that T^* maps some bounded subset of Z^* onto K.*

PROOF. Define T_0: $X \to C(K)$ to be the evaluation operator. By Theorem 1.12 T_0 maps bounded sets into GSP sets. Therefore, there exist T_1, T_2 and Z as in Theorem 1.14. The desired operator is T_1. Clearly, we may assume that T_1 has dense range: this also follows from the construction given in [3].

COROLLARY 1.17. *Let $K \subseteq X^*$ be a weak* compact, convex RNP set. Then K is affinely homeomorphic (in the weak* topologies) to a weak* compact convex subset of a conjugate Banach space with RNP.*

2. Differentiability results. Let U be an open, convex subset of the Banach space X, φ: $U \to \mathbf{R}$ a continuous function.

DEFINITION 2.1. φ is Gateaux (or weak) differentiable at $x_0 \in U$ if there exists $x_0^* \in X^*$ such that for all $x \in X$,

$$\lim_{t \to 0} \frac{\varphi(x_0 + tx) - \varphi(x_0) - t x_0^*(x)}{t} = 0.$$

DEFINITION 2.2. φ is Fréchet differentiable at x_0 if there exists an $x_0^* \in X^*$ such that

$$\lim_{\|x\| \to 0} \frac{\|\varphi(x_0 + x) - \varphi(x_0) - x_0^*(x)\|}{\|x\|} = 0.$$

The following well-known results are the only things we shall need about differentiability. Proofs can be found from at least as long ago as Smulyan [19]–[22] and Mazur [13].

THEOREM 2.3. *If U is a proper, open, convex subset of X containing the origin, then x_0 in X (strongly) exposes U^0 if and only if p, the Minkowski functional of U, is (Fréchet) Gateaux differentiable at x_0.*

THEOREM 2.4. *Let* $\varphi: X \to R$ *be continuous and convex. Assume that* $\varphi(0) < 0$ *and* $\varphi(x_0) \neq 0$. *Then, the following are equivalent.*

(1) φ *is (Fréchet) Gateaux differentiable at* x_0;

(2) *if* $G = \{(x, t): t \geqslant \varphi(x)\}$ *then* $(x_0, \varphi(x_0))$ *(strongly) exposes* G^0; *and*

(3) *the Minkowski functional of G is (Fréchet) Gateaux differentiable at* $(x_0, \varphi(x_0))$.

DEFINITION 2.5. A Banach space X is an (weak) Asplund space if every continuous, convex $\varphi: X \to \mathbf{R}$ is (Gateaux) Fréchet differentiable on a dense G_δ subset of X.

We shall also need the following result of Phelps (see **[15]** and **[4]**).

PROPOSITION 2.6. *Suppose* x_1^*, x_2^* *are in* X^*, $\|x_1^*\| = \|x_2^*\| = 1$, *and* $\{x: \|x\| \leqslant C$ *and* $x_1^*(x) \leqslant 0\}$ *is a subset of* $\{x: x_2^*(x) < \lambda\}$, *where* $C > 1$ *and* $0 < \lambda \leqslant (C-1)/4$; *then* $\|x_1^* - x_2^*\| \leqslant 2\lambda/C$.

The following lemma is essentially known (see **[25]** and **[2]**). The proof is a combination of the techniques of **[25]** and **[2]**, but is certainly much shorter than that of **[25]** and its references.

GEOMETRIC LEMMA 2.7. *Let D be a subset of* X^* *that is weak* compact, convex, and contains the origin. Then D is an RNP set if and only if every continuous, convex* $\varphi: X \to \mathbf{R}$, *such that*

$$\{(x, t): t \geqslant \varphi(x)\}^0 \subseteq D \times I$$

for some bounded interval I of $\mathbf{R}$, *is Fréchet differentiable on a dense* G_δ *set.*

PROOF. Suppose D is not an RNP set. Then there exists a separable space Z and an operator $T: Z \to X$ of norm one such that $T^*(D)$ is not norm separable. Hence, by repeating the construction of **[26]** we can find $\Delta \subseteq T^*(D)$ and $c > 0$ such that any two points in Δ have norm distance greater than c, and Δ in the $\sigma(Z^*, Z)$ topology is homeomorphic to the Cantor discontinuum. Let D_1 be a minimal weak* compact subset of D such that $T^*(D_1) = \Delta$. Define $p: X \to R$ by $p(x) = \sup\{x^*(x): x^* \in D_1\}$. Since p is the Minkowski functional of D_1^0 (which contains the origin as an interior point) p is continuous and convex. Suppose $p(x_0) = 1$ and p is differentiable at x_0. This means that x_0 strongly exposes D_1; that is, there exists an $a > 0$ such that $A = \{x^* \in D_1: x^*(x) > 1 - a\}$ has diameter less than $c/2$. Since $D_1 \setminus A$ is a proper, weak* compact subset of D_1, $T^*(D_1 \setminus A)$ is a proper, weak* compact subset of Δ. Hence, $\Delta \setminus T^*(D_1 \setminus A)$ has diameter greater than c, which implies that A has diameter greater than c, which is a contradiction. Thus, p is not differentiable where $p(x) > 0$, or, $p(x) - 1$ is not differentiable there also. It is easy to see that $\{(x, t): t \geqslant p(x) - 1\}^0$ is a subset of $D \times [-1, 1]$.

Conversely, suppose D is an RNP set and $\varphi: X \to \mathbf{R}$ is continuous, convex, and $\{(x, t): t \geqslant \varphi(x)\}^0$ is a subset of $D \times I$. If $\varphi(0) \geqslant 0$ then we may replace φ by $\psi(x) = \varphi(x) - \varphi(0) - 1$ and $\{(x, t): t \geqslant \psi(x)\}$ contains $\{(x, t): t \geqslant \varphi(x)\}$. Thus, we may assume $\varphi(0) = -1$. Since $D \times I$ is also an RNP set, we may apply Theorem 2.4 to reduce the problem to the following: φ is the Minkowski functional p of an open set U containing $B_X(0, 1)$ and $D = U^0$ is an RNP set. From Theorem

2.3 we have to show that the set of elements of X that strongly expose D is a dense G_δ subset of X. Let $S(x, a) = \{x^* \in D: x^*(x) > p(x) - a\}$ where $x \in X$, $a > 0$, and diam $S(x, a)$ is the diameter of $S(x, a)$. Let $A_m = \{x \in X$: there exists $a > 0$ such that diam $S(x, a) < 1/m\}$. Each A_m is open and $\bigcap_{m=1}^{\infty} A_m$ is exactly the set of strongly exposing elements of X. We shall show that each A_m is dense in X. Suppose there exist x_0 in X, $0 < \eta < 1$, and a positive integer m such that for any x with $\|x - x_0\| \leqslant \eta$ then diam $S(x, a) \geqslant 1/m$ for all $a > 0$. We may assume $p(x_0) > 0$ and $\|x_0\| = 1$ (if $p(x) = 0$ for x in $B_X(x_0, \eta)$ then p is differentiable there). Choose any $a_0 > 0$ and let C_0 be a number greater than $8/\eta$ and $\sup\{\|x^*\|: x^* \in D\}$.

$$B = \{x^* \in X^*: \|x^*\| \leqslant C_0^2 \text{ and } x^*(x_0) \leqslant p(x_0) - a_0\}.$$

Choose any x_0^* in D. We shall show that D is not a subset of

$$c\big(B \cup \big(\bar{B}(x_0^*, 1/4m) \cap D\big)\big).$$

Suppose there exist $b^* \in B$, $x^* \in D$ with $\|x^* - x_0^*\| \leqslant 1/4m$ and t with $0 \leqslant t \leqslant 1$ such that

$$\begin{aligned} p(x_0) - a_0/8mC_0^2 &< (tb^* + (1 - t)x^*)(x_0) \\ &\leqslant t(p(x_0) - a_0) + (1 - t)p(x_0). \end{aligned}$$

Thus $t < 1/8mC_0^2$. Also,

$$\begin{aligned} \|(tb^* + (1 - t)x^*) - x_0^*\| &\leqslant \|x^* - x_0^*\| + t(\|b^*\| + \|x_0^*\|) \\ &\leqslant 1/4m + (C_0^2 + C_0)/(8mC_0^2). \end{aligned}$$

Therefore, diam $S(x_0, a_0/8mC_0^2)$ is less than $1/m$, which is a contradiction. By the Hahn-Banach theorem there exist x_1 in X and $a_1 > 0$ such that $\|x_1\| = 1$ and

$$p(x_1) > a_1 + \sup\{x^*(x_1): x^* \in B \cup (B(x_0^*, 1/4m) \cap D)\}.$$

Since $p(x_1) \leqslant C_0$ and $8 < C_0$ we have that $C_0 < (C_0^2 - 1)/4$ and from Proposition 2.6, $\|x_1 - x_0\| < 2C_0/C_0^2 < \eta/4$. Since x_0^* was an arbitrary element of D we may, for convenience, assume that x_0^* is a point of D such that $x_0^*(x_0) = p(x_0)$. By the separation theorem there exists $y_0 \in X$, $\|y_0\| = 1$, such that

$$x_0^*(y_0) - 1/4m \geqslant \sup\{x^*(y_0): x^* \in S(x_1, a_1)\}.$$

Let $u_0 = p(y_0) - x_0^*(y_0) + 1/12m$ and suppose x_0^* is in the weak* compact, convex set $c(B \cup D \setminus S(y_0, u_0))$. That is, $x_0^* = tb^* + (1 - t)x^*$ for some t, $0 \leqslant t \leqslant 1$, $b^* \in B$, and $x^* \in D$ with $x^*(y_0) \leqslant p(y_0) - u_0$. Then,

$$p(x_0) = x_0(x_0) \leqslant t(p(x_0) - a_0) + (1 - t)p(x_0) = p(x_0) - ta_0.$$

Thus, $t = 0$ and $x^*(y_0) \leqslant p(y_0) - u_0$ which contradicts the definition of u_0. Again applying the separation theorem, we have that there exists an $x_2 \in X$, $\|x_2\| = 1$, an $a_2 > 0$ such that $x_0^*(x_2) > a_2 + \sup\{x^*(x_2): x^* \in B \cup D \setminus S(y_0, u_0)\}$. From Proposition 2.6 we have that $\|x_2 - x_0\| \leqslant 2/C_0$. Also, we have $S(x_2, a_2) \subseteq S(y_0, u_0)$ and for $x^* \in S(x_1, a_1)$, $y^* \in S(x_2, a_2)$,

$$x^*(y_0) \leqslant x_0^*(y_0) - 1/4m = p(y_0) - u_0 - 1/6m < y^*(y_0) - 1/6m.$$

Replacing C_0 by $2C_0$ we repeat this construction inside $S(x_1, a_1)$ and $S(x_2, a_2)$. Let

$x_{0,0} = x_0$, $x_{1,0} = x_1$, $x_{1,1} = x_2$, and $y_{0,0} = y_0$. Thus, we obtain sequences $\{x_{n,i}\}$ and $\{y_{n,i}\}$, $n = 0, 1, \ldots$, $i = 0, \ldots, 2^{n-1}$, of norm one elements of X and positive numbers $a_{n,i}$ such that

$$\|x_{n,i} - x_{n+1,2i+j}\| \leqslant 1/2^{n-1}C_0 \quad \text{for } j = 0, 1,$$

$$\bigcup_{j=0}^{1} S(x_{n+1,2i+j}, a_{n+1,2i+j}) \subseteq S(x_{n,i}, a_{n,i}),$$

$$\sup\{x^*(y_{n,i}): x^* \in S(x_{n+1,2i}, a_{n+1,2i})\}$$
$$\leqslant \inf\{x^*(y_{n,i}): x^* \in S(x_{n+1,2i+1}, a_{n+1,2i+1})\} - 1/6m.$$

Let Y be the smallest closed, linear subspace containing $\{y_{n,i}\}$ and

$$\Delta = \left\{ x^*|_Y: x^* \in \bigcap_{n=0}^{\infty} \bigcup_{i=0}^{2^n-1} \overline{S(x_{n,i}, a_{n,i})} \right\}.$$

The separation properties of the $y_{n,i}$ show that Δ (regarded as a subset of Y^*) is not norm separable. Thus, D is not an RNP set (Proposition 1.10), which is a contradiction. Therefore, each A_m is dense.

THEOREM 2.8. *A Banach space X is an Asplund space if and only if X^* has RNP.*

PROOF. Suppose X^* has RNP and φ: $X \to \mathbf{R}$ is continuous, convex and $\varphi(0) = -1$ (so that (0, 0) is an interior point of the epigraph G of φ). Then G^0 is a weak* compact and convex subset of $X^* \times R$ which has RNP. Apply Lemma 2.7 which also yields the converse.

The proofs of Corollaries 2.9 and 2.10 are immediate from Theorems 0 and 2.8.

COROLLARY 2.9. (i) *Subspaces and quotient spaces of Asplund spaces are Asplund*;

(ii) *if Y is a subspace of X and Y and the quotient space X/Y are Asplund then X is Asplund.*

COROLLARY 2.10. *For $1 < p < +\infty$, an l_p sum of Asplund spaces is Asplund*; *a c_0 sum of Asplund spaces is Asplund.*

THEOREM 2.11. *Let T: $X \to Y$ be a continuous, linear function. Then the following are equivalent.*

(i) *$T(\overline{B}_X)$ is a GSP set*;

(ii) *$T^*(\overline{B}_{Y^*})$ has RNP*;

(iii) *T^* factors through a space W with W having RNP*;

(iv) *for any φ: $Y \to \mathbf{R}$ that is continuous and convex, φT is differentiable on a dense subset of X*; *and*

(v) *T factors through a space Z with Z being Asplund.*

THEOREM 2.12. *The class of operators satisfying* (i)–(v) *of Theorem* 2.11 *forms an operator ideal.*

This ideal is exactly those operators T such that T^*R^* is nuclear for every R integral. In [**6**] Grothendieck proved that this ideal contains the weakly compact operators.

Combining Theorem 2.11 with the factorization theorem given in **[26]** we obtain the following.

THEOREM 2.13 (DICHOTOMY THEOREM FOR THE FACTORIZATION OF OPERATORS BETWEEN BANACH SPACES). *If $T: X \to Y$ is a continuous, linear function then one and only one of the following is true.*

(i) *T factors through an Asplund space;*

(ii) *T is a factor of the Haar operator $H: l_1 \to L_\infty(\Delta, \mu)$ as given in* **[26]**.

We point out that the dichotomy is whether $T(\bar{B}_X(0, 1))$ is GSP or not. With modifications of the techniques used here, one can prove the following.

THEOREM 2.14. *Let K be a closed, bounded and convex subset of X. Then*

(i) *K is an RNP set if and only if p_D (the Minkowski functional of D^0) is differentiable on a dense G_δ subset of X for every subset D of K;*

(ii) *the weak* closure of K in X^{**} is an RNP set if and only if for every sequence $\{x_n\}$ in K, every closed, linear subspace Y of X^*, the function $q(x^*) = \limsup x^*(x_n)$ is differentiable on a dense subset of Y.*

Using the construction in **[26]** Huff and Morris **[9]** have shown that if X^* does not have RNP then there exists a norm closed, bounded and convex subset of X^* without extreme points. Combining this with our results we have

THEOREM 2.15. *A Banach space X is an Asplund space if and only if every norm closed, bounded and convex subset of X^* is the norm closed, convex hull of its extreme points.*

3. A result on weak Asplund spaces. A Banach space is said to be GSG if it is generated by a GSP set. That is, there exists a bounded (and, we may assume, closed, convex and symmetric) GSP subset A of X such that if x^* is in X^* and $x^*(x) = 0$ for all x in A, then $x^* = 0$. We remark that by Corollary 1.15 X is GSG if and only if there exists Y as an Asplund space and an operator $T: Y \to X$ with dense range.

THEOREM 3.1. *Suppose X is GSG. Then X is a weak Asplund space.*

PROOF. Let $\varphi: X \to \mathbf{R}$ be continuous and convex and $\varphi(0) = -1$.

Let $T: Y \to X$ be an operator with dense range defined on the Asplund space Y. Define $\hat{T}: Y \times \mathbf{R} \to X \times \mathbf{R}$ by $\hat{T}((y, t)) = (Ty, t)$. Of course, $Y \times \mathbf{R}$ is an Asplund space and $\hat{T}$ has dense range. Determining points of Gateaux differentiability of φ is equivalent to determining points of G-differentiability of the Minkowski functional of

$$G = \{(x, t): t \geqslant \varphi(x), x \in X\}.$$

The subset of $X \times R$ that exposes G^0 contains all $\hat{T}((x, t))$ where (x, t) strongly exposes $\hat{T}^*(G^0)$ (since T^* is one-to-one). Define $V_n = \{(x, t) \in X \times R$: there exists a slice S of G^0 by (x, t) such that diam $\hat{T}^*(S)$ is less than $1/n\}$. Clearly, V_n is open, dense and if (x, t) is in V_n so is (sx, st) for every positive s. Therefore, the intersection of V_n with the graph of φ is open and dense in the graph of φ and

$A = \bigcap_{n=1}^{\infty} V_n \cap \{(x, \varphi(x)): x \in X\}$ is a dense G_δ subset of the graph of φ. Under the obvious map the graph of φ is homeomorphic to X. Thus,

$$C = \{x \in X: (x, \varphi(x)) \in A\}$$

is a dense G_δ subset of X. The definition of the V_n's and Theorem 2.4 show that C is contained in the set of points of Gateaux differentiability of φ.

COROLLARY 3.2 (MAZUR [**13**]). *A separable Banach space is weak Asplund.*

COROLLARY 3.3(ASPLUND [**1**]). *A WCG Banach space is weak Asplund.*

Corollaries 3.2 and 3.3 follow from [**3**]; a reflexive space is an Asplund space.

Also, we give, as another application of our results in the theory of Banach spaces, the following extensions of results of Johnson and Hagler [**7**].

THEOREM 3.4. *Let $K \subseteq X^*$ be a weak* compact, RNP set (convex or not). Then every sequence in K has a weak* converging subsequence.*

From Corollary 1.17 we may assume X^* has RNP. In fact, we can prove the following.

THEOREM 3.5. *Let X be weak Asplund, and $\{x_k^*\}$ a bounded sequence in X^*. Then $\{x_k^*\}$ has a weak* converging subsequence.*

PROOF. Let A_n be the weak* closure of $\{x_k^*: k \geqslant n\}$ and $A = \bigcap_{n=1}^{\infty} A_n$.

The set of elements of X that expose A are exactly the points of differentiability of the convex function $p(x) = \sup\{x^*(x): x^* \in A\}$ which, by hypothesis, is a dense G_δ subset of X. So there exist an $x_0 \in X$, $x_0^* \in A$ such that x_0 exposes A at x_0^*. There exists a subsequence $\{x_{k_i}^*\}$ such that $x_{k_i}^*(x_0) \to x_0^*(x_0)$. Let x_1^* be any weak* cluster point of $\{x_{k_i}^*\}$. Then $x_1^* \in A$ and $x_1^*(x_0) = x_0^*(x_0)$. Hence, $x_1^* = x_0^*$. Let B_j be the weak* closure of $\{x_{k_i}^*: i \geqslant j\}$; $\bigcap_j B_j = \{x_0\}$. A simple compactness argument shows that if $x_0^* \in U$ weak* open, then there exists a j such that $B_j \subseteq U$. This proves $x_{k_i}^*$ converges sequentially to x_0^*.

This is a very easy argument. The importance is knowing which spaces are weak Asplund spaces. The particular case of this result when X is WCG was proved by W. B. Johnson (see [**4**]).

4. Examples and problems.

1. There exists a Banach space X such that X is not WCG but X^* is WCG [**10**], [**12**]. Hence, X^* is RNP and X is Asplund (see [**24**] for a more general result).

2. For an X as in Example 1, $X \times l_1$ is GSG, but it is not Asplund and not WCG.

3. In [**18**] Rosenthal gives a subspace X of $L_1(\mu)$, μ a finite measure, such that X is not WCG. For a finite measure μ, $L_1(\mu)$ is WCG, hence GSG. By using results of [**14**] and, for example, [**23**], one can show

THEOREM. *Suppose $T: X \to L_1(\Omega, \Sigma, \mu)$ is an operator and X is Asplund. Then T is weakly compact. More generally, if $T: C(K) \to Y$ and Y has RNP, then T is weakly compact.*

This follows from a result of Pełczynski (see [14]) and the fact that if Y has RNP then no subspace of Y is isomorphic to c_0.

4. If X has an equivalent Fréchet differentiable norm, then X is Asplund. It has been known for some time [27] that if a Banach space admitted a Fréchet differentiable function with bounded support then every separable subspace of X has a separable dual–hence, X^* has RNP and from our results X is an Asplund space. This was proved in a different way in [5].

5. Pełczynski has asked the following question: suppose T: $X \to Y$ transforms bounded sets into RNP sets; does T factor through a Banach space with RNP? We have shown that the answer is yes when T is a conjugate operator–using the construction of [3].

6. If X is weak Asplund, is any closed, linear subspace of X weak Asplund?

7. Which properties of WCG spaces extend to GSG spaces? Or, at least, can one obtain known results about WCG spaces by the techniques given here, as opposed to the basic technique of proofs in WCG spaces (that of constructing equivalent norms)?

8. Does every $C(K)$, with K compact, Hausdorff and dispersed (no perfect subsets) have an equivalent Fréchet differentiable norm? Also, there is the well-known problem of characterizing Banach spaces that have equivalent Fréchet differentiable norms. The examples given in [10] and [12] make this appear difficult.

9. Suppose X and Y are separable, Y^* is not separable, X^* is separable, and T: $X \to Y$ has dense range (e.g. define T: $c_0 \to l_1$ by $T(t_i) = (2^{-i}t_i)$). Then T^* is an affine homeomorphism (in the weak* topologies) of the unit ball of Y^* into X^*, but the unit ball of Y^* is not an RNP set but its image under T^* is an RNP set. Thus, the property of weak* compact sets being RNP sets is not invariant under weak* affine homeomorphisms.

10. Is there a characterization of weak* sequential compactness of X^* in terms of some differentiability criteria of X? Probably not, but an exact determination of the relationship between these two concepts might be useful in constructing examples.

Bibliography

1. E. Asplund, *Fréchet differentiability of convex functions*, Acta Math. **121** (1968), 31–47.

2. J. Bourgain, *Strongly exposed points in weakly compact convex sets in Banach sets*, Proc. Amer. Math. Soc. **58** (1976), 197–200.

3. W. J. Davis, T. Figiel, W. B. Johnson and A. Pełczynski, *Factoring weakly compact operators*, J. Funct. Anal. **17** (1974), 311–327.

4. J. Diestel, *Geometry of Banach spaces–selected topics*, Lecture Notes in Math., vol. 485, Springer-Verlag, Berlin and New York, 1975.

5. I. Ekeland and G. Lebourg, *Generic Fréchet differentiability and perturbed optimization problems in Banach spaces*, Trans. Amer. Math. Soc. **224** (1976), 193–216.

6. A. Grothendieck, *Produits tensoriels et espaces nucleaires*, Mem. Amer. Math. Soc. No. 16 (1955).

7. J. Hagler and W. B. Johnson, *On Banach spaces whose dual balls are not weak* sequentially compact*, Israel J. Math. **28** (1977), 325–330.

8. R. Haydon, *An extreme point criterion for separability of a dual space, and a new proof of a theorem of Corson*, Quart. J. Math. Oxford Ser. (2) **11** (1976), 379–385.

9. R. E. Huff and P. D. Morris, *Dual spaces with the Krein-Milman property have the Radon-Nikodym property*, Proc. Amer. Math. Soc. **49** (1975), 104–108.

10. W. B. Johnson and J. Lindenstrauss, *Some remarks on weakly compactly generated Banach spaces*, Israel J. Math. **17** (1974), 219–230.

11. J. Lindenstrauss and A. Pełczynski, *Absolutely summing operators in $\mathcal{L}_p$-spaces and their applications*, Studia Math. **29** (1969), 275–326.

12. J. Lindenstrauss and C. Stegall, *Examples of separable spaces which do not contain l_1 and whose duals are nonseparable*, Studia Math. **54** (1975), 81–105.

13. S. Mazur, *Über konvexe Mengen in linearen normierten Räumen*, Studia Math. **4** (1933), 70–84.

14. A. Pełczynski and M. I. Kadec, *Bases, lacunary sequences and complemented subspaces in the spaces L_p*, Studia Math. **21** (1962), 161–176.

15. R. R. Phelps, *Support cones in Banach spaces and their applications*, Advances in Math. **13** (1974), 1–19.

16. ______, *Dentability and extreme points*, J. Funct. Anal. **17** (1974), 78–90.

17. R. R. Phelps and I. Namioka, *Banach spaces which are Asplund spaces*, Duke Math. J. **42** (1975), 735–750.

18. H. P. Rosenthal, *The heredity problem for weakly compactly generated Banach spaces*, Compositio Math. **28** (1974), 83–111.

19. V. Smulian, *On some geometrical properties of the unit sphere in the space of type (B)*, Math. USSR-Sb. **6** (1939), 77–94.

20. ______, *Sur la structure de la sphere unitaire dans l'espace de Banach*, Math. USSR-Sb. **9** (1941), 545–561.

21. ______, *Sur la dérivabilité de la norme dans l'espace de Banach*, C. R. (Doklady) Acad. Sci. URSS (N.S.) **27** (1940), 643–648.

22. ______, *On some geometrical properties of the sphere in a space of the type (B)*, C. R. (Doklady) Acad. Sci. URSS (N.S.) **24** (1939), 648–653.

23. C. Stegall, *Banach spaces whose duals contain $l_1(\Gamma)$ with applications to the study of conjugate $L_1(\mu)$-spaces*, Trans. Amer. Math. Soc. **176** (1973), 463–477.

24. ______, *A theorem of Haydon and its applications*, Séminaire Maurey-Schwartz, Paris, 1975-76.

25. ______, *The duality between Asplund spaces and spaces with the Radon-Nikodym property*, Israel J. Math. **29** (1978), 408–412.

26. ______, *The Radon-Nikodym property in conjugate Banach spaces*, Trans. Amer. Math. Soc. **206** (1975), 213–223.

27. F. B. Leach and J. H. M. Whitfield, *Differentiable functions and rough norms on Banach spaces*, Proc. Amer. Math. Soc. **33** (1972), 120–126.

SONDERFORSCHUNGSBEREICH 72, INSTITUT FÜR ANGEWANDTE MATHEMATIK DER UNIVERSITÄT BONN, D-5300 BONN, WEST GERMANY

MATHEMATISCHES INSTITUT DER UNIVERSITÄT ERLANGEN-NÜRNBERG, D-8520 ERLANGEN, WEST GERMANY

Current address: Institut für Mathematik, Johannes Kepler Universität, Altenbergerstrasse 69, A-4045 Linz, Austria